Linear Programming Examples

Note: All the examples are taken from chapter 29 of CLRS 3^{rd} edition.

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Summary of LP:

- With LP, we want to max or min an objective function subject to various constraints.

Note: Both the obj function and constraints must be linear.

- The feasible region in a LP is the set of all possible feasible solns.

- A feasible soln to a LP is a soln that satisfies all constraints.

- An opt soln to a LP is a feasible soln with the largest/smallest obj function value.

Note: The feasible region must be convex.

Note: An opt soln must be one of the vertices of the feasible region.

Proof:

Start at some point, x, in the feasible region and choose a direction. If you go both ways on that direction, one of the 2 paths must not decrease the obj function. We can keep going until we hit vertices.

E.g.

One of the 2 directions must not decrease the obj function.

- In Standard form of LP, we have:

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} \quad 1 \le i \le m, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\left.\begin{aligned} \text{Max} \quad & c^T x \\ & = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \end{aligned}\right\} \text{Objective function}$$

Subject to

$$m \text{ constraints} \left\{\begin{aligned} & a_1^T x \le b_1 \rightarrow a_{11} x_1 + \dots + a_{1n} x_n \le b_1 \\ & a_2^T x \le b_2 \rightarrow a_{21} x_1 + \dots + a_{2n} x_n \le b_2 \\ & \quad \vdots \\ & a_m^T x \le b_m \rightarrow a_{mn} x_1 + \dots + a_{mn} x_n \le b_m \end{aligned}\right.$$

$$n \text{ constraints} \longrightarrow x \ge 0$$

If a constraint uses $\ge$ instead of $\le$, we can do:

$a^T x \ge b \rightarrow -a^T x \le -b$

If a constraint uses $=$ instead of $\le$, we can do:

$a^T x = b \rightarrow a^T x \le b$ and $a^T x \ge b$

$\rightarrow a^T x \le b$ and $-a^T x \le -b$

If we're asked to min the obj func, we can max its negative:

$\text{Min } c^T x \rightarrow \text{Max } -c^T x$

If a var, x, is unconstrained, we can replace x by 2 vars x' and x'' s.t. we replace each occurrence of x with x' − x'' and set $x' \geq 0$, $x'' \geq 0$.

E.g. 1 Convert the below LP to Standard Form

$$
\begin{aligned}
\text{Min} \quad & -2x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 = 7 \\
& x_1 - 2x_2 \leq 4 \\
& x_1 \geq 0
\end{aligned}
$$

Soln:

The issues are:

1. Min the obj function
2. x_2 is not constrained
3. $x_1 + x_2 = 7$

To deal with x_2, we'll create x_2' and x_2'' and replace x_2 with $x_2' - x_2''$ and add $x_2' \geq 0$, $x_2'' \geq 0$

$$
\begin{aligned}
\text{Min} \quad & -2x_1 + 3(x_2' - x_2'') \\
\text{s.t.} \quad & x_1 + x_2' - x_2'' = 7 \\
& x_1 - 2(x_2' - x_2'') \leq 4 \\
& x_1, x_2', x_2'' \geq 0
\end{aligned}
$$

To deal with 3, I'll replace
$x_1 + x_2' - x_2'' = 7$ with $x_1 + x_2' - x_2'' \leq 7$ and
$-x_1 - x_2' + x_2'' \leq -7$

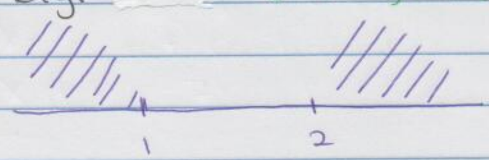
To deal with 1, I'll replace the obj func with its negative equivalent form.

The final result is:

$$\begin{aligned} \text{Max} \quad & 2x_1 - 3x_2' + 3x_2'' \\ \text{s.t.} \quad & x_1 + x_2' - x_2'' \leq 7 \\ & -x_1 - x_2' + x_2'' \leq -7 \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{aligned}$$

- An LP doesn't always have an opt soln. It can fail for 2 reasons:

1. It is infeasible. I.e. $\{x \mid Ax \leq b\} = \emptyset$
   E.g. $\{x_1 \leq 1, -x_1 \leq -2\}$



← Can't happen

2. It is unbounded.
   E.g. Max x_1 subject to $x_1 \geq 0$

- The Simplex Algo states:

Simplex Algo
let v be any vertex of the feasible region.
while there's a neighbour v' of v with a better obj value:
    set v to v'

To implement this, we'll need to work with the Slack Form of LP.

Standard Form		Slack Form
Max $c^T x$		$Z = c^T x$
s.t. $Ax \le b$	$\longrightarrow$	$S = b - Ax$
$x \ge 0$		$x, S \ge 0$

E.g. 1 Convert the below Standard Form to Slack Form

Max $2x_1 - 3x_2 + 3x_3$

s.t.
$$x_1 + x_2 - x_3 \le 7$$
$$-x_1 - x_2 + x_3 \le -7$$
$$x_1 - 2x_2 + 2x_3 \le 4$$
$$x_1, x_2, x_3 \ge 0$$

Soln: Non-basic var

$$Z = 2x_1 - 3x_2 + 3x_3$$

Basic Var
$$x_4 = 7 - x_1 - x_2 + x_3$$
$$x_5 = -7 + x_1 + x_2 - x_3$$
$$x_6 = 4 - x_1 + 2x_2 - 2x_3$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

E.g. 2 Given the below Slack Form, use the Simplex Algo to find an opt soln.

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$
$$x_1, \ldots, x_6 \geq 0$$

Soln:

Step 1:

We start at a feasible vertex.
For now, assume that $b \geq 0$.
In this case, $x = 0$ is a feasible vertex.
In Slack Form, this means setting the non-basic vars to 0.

To increase the value of z, choose a non-basic var with a positive coefficient (this is called the entering var) and see how much we can increase its value without violating any constraints.

I'll choose x_1.

$-x_1 = x_4 - 30$	$2x_1 = 24 - x_5$	$4x_1 = 36 - x_6$
$x_1 = 30 - x_4$	$x_1 = 12 - \frac{x_5}{2}$	$x_1 = 9 - \frac{x_6}{4}$
≤ 30	≤ 12	≤ 9 tightest bound

Note: x_2 and $x_3 = 0$ from above and $x_4, x_5, x_6 \geq 0$.

Now, we'll solve the tightest bound for the non-basic var.

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Now, we'll substitute the entering var (called pivot) in other eqns.
Now, X_1 is basic and X_6 is non-basic.
X_6 is called the leaving var.

I'll replace X_1 with $9 - \frac{X_2}{4} - \frac{X_3}{2} - \frac{X_6}{4}$

$$Z = 3\left(9 - \frac{X_2}{4} - \frac{X_3}{2} - \frac{X_6}{4}\right) + X_2 + 2X_3$$
$$X_1 = 9 - \frac{X_2}{4} - \frac{X_3}{2} - \frac{X_6}{4}$$
$$X_4 = 30 - \left(9 - \frac{X_2}{4} - \frac{X_3}{2} - \frac{X_6}{4}\right) - X_2 - 3X_3$$
$$X_5 = 24 - 2\left(9 - \frac{X_2}{4} - \frac{X_3}{2} - \frac{X_6}{4}\right) - 2X_2 - 5X_3$$

$X_1, \dots, X_6 \geq 0 \rightarrow$

$\downarrow$

$$Z = 27 + \frac{X_2}{4} + \frac{X_3}{2} - \frac{3X_6}{4}$$
$$X_1 = 9 - \frac{X_2}{4} - \frac{X_3}{2} - \frac{X_6}{4}$$
$$X_4 = 21 - \frac{3X_2}{4} - \frac{5X_3}{2} + \frac{X_6}{4}$$
$$X_5 = 6 - \frac{3X_2}{2} - 4X_3 + \frac{X_6}{2}$$
$$X_1, \dots, X_6 \geq 0$$

We keep repeating this process until there are no entering var.

Step 2:

I'll use X_3 as the next entering var.

$X_1 = 9 - \frac{X_3}{2}$	$X_4 = 21 - \frac{5X_3}{2}$	$X_5 = 6 - 4X_3$
$2X_1 = 18 - X_3$	$\frac{2X_4}{5} = \frac{42}{5} - X_3$	$X_3 = \frac{6}{4} - \frac{X_5}{4}$
$X_3 = 18 - 2X_1$	$X_3 = \frac{42}{5} - \frac{2X_4}{5}$	$\leq \frac{6}{4}$ Tightest Bound
≤ 18	$\leq \frac{42}{5}$	

$$X_3 = \frac{6}{4} - \frac{3X_2}{8} - \frac{X_5}{4} + \frac{X_6}{8}$$

$$Z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$
$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$
$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$
$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$
$$x_1, \dots, x_6 \geq 0$$

Step 3:

I'll use x_2 as the entering var.

$x_1 = \frac{33}{4} - \frac{x_2}{16}$	$x_3 = \frac{3}{2} - \frac{3x_2}{8}$	$x_4 = \frac{69}{4} + \frac{3x_2}{16}$
$x_2 = 132 - 16x_1$	$x_2 = 4 - \frac{8}{3}x_3$	$x_2 = 92 - \frac{16x_4}{3}$
≤ 132	≤ 4	Can't use

Tightest Bound

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$

$$Z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$
$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}$$
$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$
$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}$$
$$x_1, \dots, x_6 \geq 0$$

Take the basic feasible soln ($x_3 = x_5 = x_6 = 0$) and that gives an opt value of $z = 28$.

In the opt soln, $x_1 = 8$, $x_2 = 4$, $x_3 = 0$.

- The dual LP states that if

$$\left.\begin{array}{ll} \text{Max} & c^T x \\ \text{Subject to} & Ax \le b \\ & x \ge 0 \end{array}\right\} \text{Primal LP}$$

is an LP in Standard Form, then its dual LP is

$$\left.\begin{array}{ll} \text{Min} & b^T y \\ \text{Subject to} & A^T y \ge c \\ & y \ge 0 \end{array}\right\} \text{Dual LP}$$

E.g. Convert the below Standard Form to Dual Form

$$\begin{array}{ll} \text{Max} & 1x_1 + 2x_2 + 3x_3 + 4x_4 \\ \text{s.t.} & 8x_1 + 9x_2 + 10x_3 + 11x_4 \le 5 \\ & 12x_1 + 13x_2 + 14x_3 + 15x_4 \le 6 \\ & 16x_1 + 17x_2 + 18x_3 + 19x_4 \le 7 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

Soln:

$c = [1, 2, 3, 4]$, $b = [5, 6, 7]$

$$A = \begin{bmatrix} 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 \end{bmatrix}$$

Dual LP:

$$\begin{array}{ll} \text{Min} & 5y_1 + 6y_2 + 7y_3 \\ \text{s.t.} & 8y_1 + 12y_2 + 16y_3 \ge 1 \\ & 9y_1 + 13y_2 + 17y_3 \ge 2 \\ & 10y_1 + 14y_2 + 18y_3 \ge 3 \\ & 11y_1 + 15y_2 + 19y_3 \ge 4 \end{array}$$

- The dual is formed by:

1. Having 1 var for each constraint of the primal, not counting the $x \geq 0$ constraints.
2. Having 1 constraint for each var of the primal, plus the $x \geq 0$ constraints.

- The weak duality theorem for any primal feasible x and dual feasible y, $c^T x \leq y^T b$.

Proof:

$$
\begin{aligned}
c^T x &\leq (y^T A) x \\
&\leq (y^T) A x \\
&\leq y^T b
\end{aligned}
$$

Question 29.1-4:

$$\text{Max} \quad -2x_1' + 2x_1'' - 7x_2 - x_3$$
$$\text{s.t.} \quad x_1' - x_1'' - x_3 \le 7$$
$$-x_1' + x_1'' + x_3 \le -7$$
$$-3x_1' + 3x_1'' - x_2 \le -24$$
$$x_1', x_1'', x_2, -x_3 \ge 0$$

Question 29.1-5

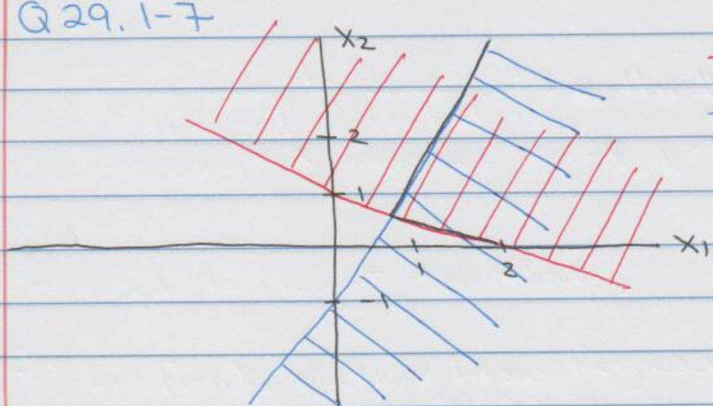
$$z = 2x_1 - 6x_3$$
$$x_4 = 7 - x_1 - x_2 + x_3$$
$$x_5 = -8 + 3x_1 - x_2$$
$$x_6 = 0 - x_1 + 2x_2 + 2x_3$$
$$x_1, \ldots, x_6 \ge 0$$

Basic vars: x_4, x_5, x_6
Non-basic vars: x_1, x_2, x_3

Q29.1-6

We have $x_1 + x_2 \le 2$ and $-2x_1 - 2x_2 \le -10$.
The 2nd inequality is equivalent to $x_1 + x_2 \ge 5$.
$\therefore$ The LP is infeasible.

Q29.1-7



$-x_1 - 2x_2 \le -2$
$-2x_1 + x_2 \le -1$

The area where the red and blue lines intersect, provided that $x_1, x_2 \ge 0$ is the feasible region. We can see its unbounded.

Q 29.3-5:

$Z = 18x_1 + 12.5x_2$
$x_3 = 20 - x_1 - x_2$
$x_4 = 12 - x_1$
$x_5 = 16 - x_2$
$x_1, \ldots, x_5 \geq 0$

Choose x_1

$x_1 = 20 - x_3$	$x_1 = 12 - x_4$
≤ 20	≤ 12
	Tightest Bound

$x_1 = 12 - x_4$

$Z = 216 - 18x_4 + 12.5x_2$
$x_1 = 12 - x_4$
$x_3 = 8 - x_2 + x_4$
$x_5 = 16 - x_2$
$x_1, \ldots, x_5 \geq 0$

Choose x_2

$x_2 = 8 - x_3$	$x_2 = 16 - x_5$
≤ 8	≤ 16
Tightest Bound	

$Z = 316 - 18x_4 - 12.5x_3$
$x_1 = 12 - x_4$
$x_2 = 8 - x_3 + x_4$
$x_5 = 8 + x_3 - x_4$

$Z = 316$
$x_1 = 12$
$x_2 = 8$